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Projective differential geometry of higher reductions of the two-dimensional Dirac equation

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Abstract

We investigate reductions of the two-dimensional Dirac equation imposed by the requirement of the existence of a differential operator D_n of order n mapping its eigenfunctions to adjoint eigenfunctions. For first order operators these reductions (and multicomponent analogs thereof) lead to the Lame equations descriptive of orthogonal coordinate systems. Our main observation is that nth order reductions coincide with the projective-geometric 'Gauss–Codazzi' equations governing special classes of line congruences in the projective space P^{2n-1}, which is the projectivised kernel of D_n. In the second order case this leads to the theory of W-congruences in P^3 which belong to a linear complex, while the third order case corresponds to isotropic congruences in P^5. Higher reductions are compatible with odd order flows of the Davey–Stewartson hierarchy. All these flows preserve the kernel D_n, thus defining nontrivial geometric evolutions of line congruences.

Multicomponent generalizations are also discussed. The correspondence between geometric picture and the theory of integrable systems is established; the definition of the class of reductions and all geometric objects in terms of the multicomponent KP hierarchy is presented. Generating forms for reductions of arbitrary order are constructed.

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1. Introduction

The two-dimensional Dirac equation

$$\partial_x \Psi_2 = \beta \Psi_1, \qquad \partial_y \Psi_1 = \gamma \Psi_2, \tag{1}$$

plays an important role in differential geometry and mathematical physics, arising, in particular, as the Lax operator of the Davey–Stewartson (DS) hierarchy. Our main goal is to investigate the class of its reductions (that is, differential constraints on the potentials $\beta(x, y)$ and $\gamma(x, y)$, possibly nonlocal [23]), which are compatible with odd order flows of the DS hierarchy (the modified Veselov–Novikov system being the first nontrivial flow among them [2]). The simplest representative of this class of reductions (in one-component case) is probably the BKP hierarchy [15]. Reductions describing Veselov–Novikov equation and modified Veselov–Novikov equation [2,14,28] also belong to the class we study. In terms of the $\bar\partial$-dressing method these reductions manifest themselves as special linear conditions on the kernel of the nonlocal $\bar\partial$-problem, and it was shown in [33] that Lame equations describing N-orthogonal coordinate systems can be obtained in this way. In the work [34] a general class of conditions on the kernel of the nonlocal $\bar\partial$-problem was described.

The general approach we adopt here is to require the existence of a linear differential operator D_n of order n which maps solutions of (1) to solutions of the adjoint problem

$$\partial_x \Psi_2^* = \gamma \Psi_1^*, \qquad \partial_y \Psi_1^* = \beta \Psi_2^*. \tag{2}$$

In this direct form the method was probably first proposed in [25]. Explicitly, we require

$$\Psi_1^* = c_1 \partial_x^n \Psi_1 + \cdots, \qquad \Psi_2^* = c_2 \partial_y^n \Psi_2 + \cdots, \tag{3}$$

where $c_i = \text{const}$ and dots denote an arbitrary linear combination of the terms $\partial_x^k \Psi_1$ and $\partial_y^k \Psi_2$, $k = 0, 1, \ldots, n-1$. Notice that the requirement of Ψ_1^*, Ψ_2^* being adjoint eigenfunctions imposes strong constraints on the coefficients in (3), specifying them almost uniquely up to a certain natural equivalence. The zero order case

$$\Psi_1^* = c_1 \Psi_1, \qquad \Psi_2^* = c_2 \Psi_2,$$

implies the well-known reduction

$$c_1 \gamma = c_2 \beta.$$

In the case of first order operators

$$\Psi_1^* = c_1 \partial_x \Psi_1 + c_2 \beta \Psi_2, \qquad \Psi_2^* = c_2 \partial_y \Psi_2 + c_1 \gamma \Psi_1,$$

one obtains the reduction

$$c_1 \gamma_x + c_2 \beta_y = 0.$$

Our main observation is that reductions of higher order are intimately connected with projective differential geometry. Moreover, the venue of this geometry is the projectivised kernel P^{2n-1} of the operator D_n. For instance, the case of D_2 is characterized by

$$\Psi_1^* = c_1(\partial_x^2\Psi_1 - U\Psi_1) + c_2(\beta\partial_y\Psi_2 - \beta_y\Psi_2),$$
$$\Psi_2^* = c_2(\partial_y^2\Psi_2 - V\Psi_2) + c_1(\gamma\partial_x\Psi_1 - \gamma_x\Psi_1), \tag{4}$$

where the potentials β, γ and the 'nonlocalities' U, V satisfy the reduction equations

$$c_1(\gamma_{xx} - U\gamma) = c_2(\beta_{yy} - V\beta), \qquad U_y = 3\gamma_x\beta + \gamma\beta_x, \qquad V_x = 3\beta_y\gamma + \beta\gamma_y. \tag{5}$$

The four-dimensional kernel of D_2 is defined by the equations

$$\partial_x\Psi_2 = \beta\Psi_1, \qquad \partial_y\Psi_1 = \gamma\Psi_2, \qquad \partial_x^2\Psi_1 = U\Psi_1 - \frac{c_2}{c_1}(\beta\partial_y\Psi_2 - \beta_y\Psi_2),$$
$$\partial_y^2\Psi_2 = V\Psi_2 - \frac{c_1}{c_2}(\gamma\partial_x\Psi_1 - \gamma_x\Psi_1), \tag{6}$$

which, as we point out in Section 2, coincide with the standard Wilczynski-type equations of a W-congruence in P^3 with two focal surfaces Ψ_1 and Ψ_2, referred to conjugate coordinates x and y. Moreover, this congruence lies in a linear complex which can be constructed as follows. Let us introduce a potential S by the formulae

$$S_x = \Psi_1\Phi_1^* - \Psi_1^*\Phi_1, \qquad S_y = \Psi_2\Phi_2^* - \Psi_2^*\Phi_2,$$

which explicitly integrate to a skew-symmetric bilinear form S on the space of solutions of (1)

$$S(\Psi, \Phi) = c_1(\Psi_1\partial_x\Phi_1 - \Phi_1\partial_x\Psi_1) + c_2(\Psi_2\partial_y\Phi_2 - \Phi_2\partial_y\Psi_2). \tag{7}$$

The restriction of S to the kernel of D_2 is the invariant skew-symmetric bilinear form defining a linear complex of lines in P^3 (recall that a linear complex in P^3 is given by one linear equation in Plücker coordinates, that is, by a 4×4 skew-symmetric matrix). As we demonstrate in Section 2, the congruence (6) lies in the linear complex defined by S. The reduction equation (5) are nothing but the corresponding projective 'Gauss–Codazzi' equations.

Similarly, the general form of D_3 is

$$\Psi_1^* = c_1(\partial_x^3\Psi_1 - 2V\partial_x\Psi_1 - V_x\Psi_1) + c_2(\beta\partial_y^2\Psi_2 - \beta_y\partial_y\Psi_2 + (\beta_{yy} - 2V\beta)\Psi_2),$$
$$\Psi_2^* = c_2(\partial_y^3\Psi_2 - 2W\partial_y\Psi_2 - W_y\Psi_2) + c_1(\gamma\partial_x^2\Psi_1 - \gamma_x\partial_x\Psi_1 + (\gamma_{xx} - 2W\gamma)\Psi_1), \tag{8}$$

where the potentials β, γ and the nonlocalities V, W satisfy the third order reduction equations

$$c_1(\beta_{yyy} - 2\beta_y W - \beta W_y) + c_2(\gamma_{xxx} - 2\gamma_x V - \gamma V_x) = 0,$$
$$W_x = 2\gamma\beta_y + \beta\gamma_y, \qquad V_y = 2\beta\gamma_x + \gamma\beta_x. \tag{9}$$

The kernel of D_3 is six-dimensional, defined by the equations

$$\begin{aligned}
&\partial_x\Psi_2 = \beta\Psi_1, \qquad \partial_y\Psi_1 = \gamma\Psi_2,\\
&\partial_x^3\Psi_1 = 2V\partial_x\Psi_1 + V_x\Psi_1 - \frac{c_2}{c_1}(\beta\partial_y^2\Psi_2 - \beta_y\partial_y\Psi_2 + (\beta_{yy} - 2V\beta)\Psi_2),\\
&\partial_y^3\Psi_2 = 2W\partial_y\Psi_2 + W_y\Psi_2 - \frac{c_1}{c_2}(\gamma\partial_x^2\Psi_1 - \gamma_x\partial_x\Psi_1 + (\gamma_{xx} - 2W\gamma)\Psi_1),
\end{aligned} \tag{10}$$

which, as we demonstrate in Section 3, also give rise to a line congruence with the two focal surfaces Ψ_1 and Ψ_2. Moreover, in this case one can introduce a potential S by the formulae

$$S_x = \Psi_1\Psi_1^*, \qquad S_y = \Psi_2\Psi_2^*,$$

which explicitly integrate to a quadratic form S on the space of solutions of (1)

$$S(\Psi,\Psi) = c_1(\Psi_1\partial_x^2\Psi_1 - \tfrac{1}{2}(\partial_x\Psi_1)^2 - V\Psi_1^2) + c_2(\Psi_2\partial_y^2\Psi_2 - \tfrac{1}{2}(\partial_y\Psi_2)^2 - W\Psi_2^2). \tag{11}$$

In the case $c_1 = -c_2 = 1$, the restriction of S to the kernel of D_3 defines the invariant symmetric scalar product of the signature $(3, 3)$. The corresponding congruence is isotropic with respect to S, thus coinciding with the Plücker image of a surface in P^3. Reduction equation (9) are nothing but the projective 'Gauss–Codazzi' equations of surfaces in P^3. The necessary information on Wilczynski's approach to surfaces in P^3 and their Plücker images in P^5 is included in Appendix A.

In the case $c_1 = c_2 = 1$, the linear system (10) is descriptive of a surface in Lie sphere geometry (in hexaspherical representation), Eq. (9) being the corresponding Lie-geometric 'Gauss–Codazzi' equations [10]. The quadratic form S is in this case of the signature $(4, 2)$, defining the Lie-invariant scalar product on the kernel of D_3.

All higher reductions are invariant under Bäcklund transformations of the Darboux–Levi type [22]. In the first order case this was pointed out in [25]. In the case of third order reductions the corresponding Darboux–Levi transformations coincide with transformations W (Appendix B).

This approach obviously carries over to the multicomponent linear problem

$$\partial_i\Psi_j = \beta_{ij}\Psi_i, \tag{12}$$

$i, j = 1, \dots, N$; here the potentials β_{ij} must satisfy the compatibility conditions $\partial_k\beta_{ij} = \beta_{ik}\beta_{kj}$. Introducing the adjoint linear problem

$$\partial_i\Psi_j^* = \beta_{ji}\Psi_i^*, \tag{13}$$

we define nth order reductions by requiring the existence of an operator D_n of order n, mapping eigenfunctions Ψ_i to the corresponding adjoint eigenfunctions

$$\Psi_i^* = c_i\partial_i^n\Psi_i + \cdots, \quad c_i = \text{const.} \tag{14}$$

In the zero order case we have

$$\Psi_i^* = c_i\Psi_i,$$

implying the familiar 'Egorov' reduction

$$c_j\beta_{ij} = c_i\beta_{ji}.$$

First order reductions are defined by

$$\Psi_i^* = c_i \partial_i \Psi_i + \sum_{k \neq i} c_k \beta_{ik} \Psi_k,$$

leading to the 'Lame' equations

$$c_i \partial_i \beta_{ji} + c_j \partial_j \beta_{ij} + \sum_{k \neq i,j} c_k \beta_{ik} \beta_{jk} = 0,$$

which are descriptive of N-orthogonal coordinate systems. In this form, first order reductions of the system (12) were discussed by Schief [25], see also [33,35] for an alternative approach.

In the multicomponent situation, all higher reductions also have a clear differential-geometric interpretation, governing special Laplace sequences (of congruences lying in linear complexes, isotropic congruences, etc., see Sections 2 and 3 for explicit formulae and geometric discussion).

The class of reductions of the form (3) has a very natural interpretation in terms of the multicomponent KP hierarchy. In fact, this work started from the observation that 'Gauss–Codazzi' equation (9) of surfaces in P^3 belong to a special class of reductions of Davey–Stewartson hierarchy with odd times. This observation has given an impetus to the study of connections of this class of reductions with projective geometry, the first results of which are presented in this work.

Multicomponent KP hierarchy corresponds to the dynamics of the multicomponent Grassmannian defined by the loop group [17,24,26]. Let the Grassmannian point $W(\lambda)$ be realized as a space of vector functions (rows of length N) on the unit circle $\mathbf{S}$ in the complex plane $\mathbb{C}$, and the dual point $W^*(\lambda)$ as a space of functions taking their values in the space of columns of height N

$$\oint W(\lambda) W^*(\lambda)\, \mathrm{d}\lambda = 0. \tag{15}$$

The loop group Γ^{+^N}, represented here by $N \times N$ diagonal matrix functions $g(\lambda)$ on $\mathbf{S}$, $g_{ij}(\lambda) = \delta_{ij} g_i(\lambda)$, $g_i(\lambda) \in \Gamma^+$ (i.e., functions $g_i(\lambda)$ are analytic functions in the unit disc $\mathbf{D}$ having no zeroes), defines a dynamics (deformations) on the Grassmannian

$$W(g; \lambda) = W_0(\lambda) g^{-1}(\lambda), \qquad W^*(g; \lambda) = g(\lambda) W_0^*(\lambda), \tag{16}$$

which evidently preserves the duality (15). If we introduce the standard parameterization of Γ^{+^N} in terms of an infinite number of variables ('times' of the hierarchy)

$$g_i(\lambda) = g_i(\lambda, \mathbf{x}) = \exp\left(\sum_{n=1}^{\infty} x_{i(n)} \lambda^n\right), \tag{17}$$

the points of the Grassmannian will depend on infinite number of 'times', leading to the standard picture of the hierarchy in terms of PDEs. It is easy to derive linear equation (12) and dual equation (13) in this context (see, e.g., [3]). The variables $x_i = x_{i(1)}$ correspond to the variables x_i introduced above.

The standard reduction of KP hierarchy describing, in scalar case, the nth Gelfand–Dickey hierarchy ($n = 2$ corresponds to KdV hierarchy), is the condition

$$\lambda^n W(\lambda) \subset W(\lambda), \tag{18}$$

which is preserved by the dynamics and leads to the stationarity of the point of the Grassmannian with respect to some 'time' of the hierarchy. In this sense conditions of this type always lead to dimensional reduction (from a hierarchy of $(2+1)$-dimensional integrable equations to the hierarchy of $(1+1)$-dimensional integrable equations). The important feature of the class of reductions we study is that these reductions *do not reduce the dimension*, and the corresponding integrable systems are $(2+1)$-dimensional. These reductions are consistent not with the full dynamics given by Γ^+, rather than only the dynamics connected with its subgroup defined by the condition

$$g(\lambda)g(-\lambda) = I. \tag{19}$$

In terms of the times (17), this condition just means that even times are equal to zero, and only evolution with respect to odd times is considered.

In this case, in addition to a pair of dual Grassmannian points, we also consider a dual pair $W^{*\mathrm{t}}(g, -\lambda)$ and $W^{\mathrm{t}}(g; -\lambda)$ ('t' is for transposed). It is easy to see that the dynamics (16) defined by the subgroup (19) is identical on the spaces $W(g; \lambda)$ and $W^{*\mathrm{t}}(g, -\lambda)$ ($W^*(g; \lambda)$ and $W^{\mathrm{t}}(g, -\lambda)$, respectively).

Thus, conditions of the type

$$W^{*\mathrm{t}}(g; -\lambda)F(\lambda) \subset W(g; \lambda), \tag{20}$$

where $F(\lambda)$ is a diagonal matrix, $F_{ij} = \delta_{ij}F_i(\lambda) = \delta_{ij}c_i\lambda^n$, are invariant in the dynamics, and define reductions of the initial system. This is exactly the class of reductions (3) we have introduced above, and it is a rather straightforward technical step to derive transformation (3) from the condition (20).

Solutions to multicomponent KP hierarchy can be constructed using the $\bar\partial$-dressing method [32]. The spaces $W(\mathbf{x}, \lambda)$ and $W^*(\mathbf{x}, \lambda)$ correspond in this case to spaces of solutions of the nonlocal $\bar\partial$-problem and the dual problem (see the explicit construction in [3]). The class of reductions (20) corresponds in this case to very simple conditions on the kernel of the nonlocal matrix $\bar\partial$-problem

$$R^{\mathrm{t}}(-\mu, -\lambda) = F(\mu)R(\lambda, \mu)F^{-1}(\lambda).$$

Conditions of this type were considered in [2,14,33–35].

2. Second order reductions and congruences in P^3 lying in linear complexes

2.1. Two-component case

Under the reduction conditions (5), the linear system (6)

$$\partial_x\Psi_2 = \beta\Psi_1, \qquad \partial_y\Psi_1 = \gamma\Psi_2, \qquad \partial_x^2\Psi_1 = U\Psi_1 - \frac{c_2}{c_1}(\beta\partial_y\Psi_2 - \beta_y\Psi_2),$$
$$\partial_y^2\Psi_2 = V\Psi_2 - \frac{c_1}{c_2}(\gamma\partial_x\Psi_1 - \gamma_x\Psi_1),$$

which describes the kernel of D_2, is compatible of rank 4 (indeed, the solution is completely determined by the values of $\Psi_1, \Psi_2, \partial_x\Psi_1, \partial_y\Psi_2$ at a fixed point x, y). Therefore, both $\Psi_1(x, y)$ and $\Psi_2(x, y)$ can be interpreted as four-component position vectors of two surfaces in P^3, M_1 and M_2 (written in homogeneous coordinates). The first two equations in (6) clearly imply that the parametrization x, y is conjugate, moreover, the line (Ψ_1, Ψ_2) is tangential to both M_1 and M_2 along the y- and x-directions, respectively. In other words, M_1 and M_2 are focal surfaces of the line congruence (Ψ_1, Ψ_2). Historically, the theory of line congruences (that is, two-parameter families of lines in P^3) has been one of the most popular chapters of projective differential geometry, dating back to Monge, Plücker and Kummer. In the majority of geometrical applications the main role is played by special congruences known as congruences W (named after Weingarten), which are characterized by the property that the (projective) second fundamental forms of the focal surfaces coincide. A direct computation shows that second fundamental forms of both focal surfaces M_1 and M_2 indeed coincide, both being conformal to $c_2\beta\,\mathrm{d}x^2 - c_1\gamma\,\mathrm{d}y^2$ (recall that only the conformal class of the second fundamental form is a projective invariant).

Let us introduce a skew-symmetric scalar product $\{,\}$ on the kernel of D_2 defined by the bilinear form (7)

$$\{\Psi_1, \partial_x\Psi_1\} = \frac{1}{c_1}, \qquad \{\Psi_2, \partial_y\Psi_2\} = \frac{1}{c_2} \tag{21}$$

(all other scalar products being zero). Since the two-dimensional plane spanned by Ψ_1 and Ψ_2 in the four-dimensional kernel of D_2 is clearly Lagrangian (indeed, $\{\Psi_1, \Psi_2\} = 0$), the projective line (Ψ_1, Ψ_2) belongs to the corresponding linear complex. The representation of congruences of linear complexes in the form (6) can be traced back to the works [30,31]. The compatibility conditions (5) are nothing but the corresponding projective 'Gauss–Codazzi' equations. It should be emphasized that congruences of linear complexes constitute a proper subclass of congruences W.

All odd order flows of the DS hierarchy preserve the kernel of D_2 and the skew-symmetric form S. Thus, they induce geometric evolutions of congruences of linear complexes (so that the equation of the complex stays the same in the process of evolution). Moreover, these evolutions preserve the conjugate parametrization x, y and the projectively invariant functional

$$\int \beta\gamma\,\mathrm{d}x\,\mathrm{d}y,$$

which is the first nontrivial conservation law of the DS hierarchy.

Similar integrable evolutions in conformal, affine and Lie sphere geometries were a subject of recent publications [5,11,18–20,27].

2.2. *N*-component case

In the N-component case (12), Eq. (4) take the form

$$\Psi_i^* = c_i(\partial_i^2\Psi_i - U_i\Psi_i) + \sum_{\substack{1\le p\le N,\\ p\ne i}} c_p(\beta_{ip}\partial_p\Psi_p - (\partial_p\beta_{ip})\Psi_p), \tag{22}$$

leading to the reduction

$$c_i(\partial_i^2\beta_{ji} - U_i\beta_{ji}) - c_j(\partial_j^2\beta_{ij} - U_j\beta_{ij}) + \sum_{\substack{1\le p\le N,\\ p\ne i,j}} c_p(\beta_{ip}\partial_p\beta_{jp} - \beta_{jp}\partial_p\beta_{ip}) = 0,$$

$$\partial_j U_i = 3\beta_{ij}\partial_i\beta_{ji} + \beta_{ji}\partial_i\beta_{ij}, \quad i \ne j, \tag{23}$$

which, for $N = 2$, coincides with (5) after the identification $\beta_{12} = \beta$, $\beta_{21} = \gamma$. The kernel of D_2 is defined by the linear system

$$\partial_i\Psi_j = \beta_{ij}\Psi_i, \qquad \partial_i^2\Psi_i = U_i\Psi_i - \sum_{\substack{1\le p\le N,\\ p\ne i}} \frac{c_p}{c_i}(\beta_{ip}\partial_p\Psi_p - (\partial_p\beta_{ip})\Psi_p),$$

which is compatible of rank $2N$ by virtue of the reduction equation (23). Each of the Ψ_i's can thus be regarded as a position vector of an N-dimensional submanifold M_i in P^{2N-1}, parametrized by conjugate coordinates $x_1, \dots, x_n$, $\partial_i = \partial/\partial_{x_i}$. Moreover, the line (Ψ_i, Ψ_j) is tangential to both M_i and M_j. A simple calculation shows that the second fundamental forms of M_i (notice that there should be $N-1$ thereof, where $N-1$ is the codimension of M_i in P^{2N-1}), are given by

$$c_i\beta_{pi}\,\mathrm{d}x_p^2 - c_p\beta_{ip}\,\mathrm{d}x_i^2, \quad p \ne i.$$

Each pair of focal submanifolds (say, M_i and M_j), has one second fundamental form 'in common', namely, the form $c_i\beta_{ji}\,\mathrm{d}x_j^2 - c_j\beta_{ij}\,\mathrm{d}x_i^2$. This can be interpreted as a multicodimensional analog of the W-property.

Let us introduce a potential S by the formulae

$$\partial_i S = \Psi_i\Phi_i^* - \Psi_i^*\Phi_i, \tag{24}$$

which explicitly integrate to a skew-symmetric bilinear form S on the space of solutions of (12)

$$S(\Psi, \Phi) = \sum_{i=1}^{N} c_i(\Psi_i\partial_i\Phi_i - \Phi_i\partial_i\Psi_i). \tag{25}$$

The restriction of S to the kernel of D_2 is the invariant skew-symmetric bilinear form corresponding to the skew-symmetric scalar product

$$\{\Psi_i, \partial_i\Psi_i\} = \frac{1}{c_i} \tag{26}$$

(all other scalar products being zero). Since the N-dimensional subspace spanned by Ψ_i in the $2N$-dimensional kernel of D_2 is clearly Lagrangian (indeed, $\{\Psi_i, \Psi_j\} = 0$), each of the congruences (Ψ_i, Ψ_j) belongs to one and the same linear complex defined by S.

To the best of our knowledge, the geometric object consisting of N submanifolds of codimension $N-1$ in P^{2N-1}, connected by congruences which belong to a linear complex, has not been discussed before.

3. Third order reductions and isotropic congruences in P^5

3.1. Two-component case

Under the reduction equation (9), the linear system (10)

$$\begin{aligned}
\partial_x\Psi_2 &= \beta\Psi_1, \qquad \partial_y\Psi_1 = \gamma\Psi_2,\\
\partial_x^3\Psi_1 &= 2V\partial_x\Psi_1 + V_x\Psi_1 - \frac{c_2}{c_1}(\beta\partial_y^2\Psi_2 - \beta_y\partial_y\Psi_2 + (\beta_{yy} - 2V\beta)\Psi_2),\\
\partial_y^3\Psi_2 &= 2W\partial_y\Psi_2 + W_y\Psi_2 - \frac{c_1}{c_2}(\gamma\partial_x^2\Psi_1 - \gamma_x\partial_x\Psi_1 + (\gamma_{xx} - 2W\gamma)\Psi_1),
\end{aligned}$$

describing the kernel of D_3, is compatible of rank 6 (indeed, the solution is completely determined by the values of $\Psi_1, \Psi_2, \partial_x\Psi_1, \partial_y\Psi_2, \partial_x^2\Psi_1, \partial_y^2\Psi_2$ at a fixed point x, y). Therefore, both $\Psi_1(x, y)$ and $\Psi_2(x, y)$ can be interpreted as position vectors of two surfaces in P^5, M_1 and M_2. Like in the second order case, the first two equations in (10) imply that the parametrization x, y is conjugate, moreover, the line (Ψ_1, Ψ_2) is tangential to both M_1 and M_2 along the y- and x-directions, respectively. In other words, M_1 and M_2 are focal surfaces of the line congruence (Ψ_1, Ψ_2). In the case $c_1 = -c_2 = 1$, the quadratic form S given by (11) defines an invariant symmetric scalar product of the signature (3, 3) on the kernel of D_3 (see Appendix A). Moreover, the congruence (Ψ_1, Ψ_2) is isotropic with respect to S. This implies that M_1 and M_2 are Plücker images of asymptotic tangents to a surface in P^3. The passage from a surface in P^3 to its Plücker image in P^5 is a classical projective-geometric construction discussed in detail in [1,4,12,13], see a short review in Appendix A. Linear system (10) defines the standard frame associated with the Plücker image. The reduction equation (9), which are the compatibility conditions of (10), are nothing but the projective 'Gauss–Codazzi' equations of surfaces in P^3, see (A.2).

All odd order flows of the DS hierarchy preserve the kernel of D_3 and the quadratic form S. Hence, they induce geometric evolutions of isotropic congruences (and, therefore, surfaces in P^3), preserving the parametrization x, y (which is conjugate in P^5 and asymptotic in P^3), and the projectively invariant functional

$$\int \beta\gamma \,\mathrm{d}x\,\mathrm{d}y,$$

which is the projective area.

3.2. N-component case

In the N-component case, Eq. (8) takes the form

$$\begin{aligned}
\Psi_i^* = c_i(\partial_i^3\Psi_i - 2V_i\partial_i\Psi_i - (\partial_i V_i)\Psi_i) + \sum_{\substack{1\le p\le N,\\ p\neq i}} c_p(\beta_{ip}\partial_p^2\Psi_p\\
-(\partial_p\beta_{ip})\partial_p\Psi_p + (\partial_p^2\beta_{ip} - 2V_p\beta_{ip})\Psi_p),
\end{aligned}$$

the corresponding reduction equations being

$$
\begin{aligned}
&c_i(\partial_i^3\beta_{ji} - 2V_i\partial_i\beta_{ji} - (\partial_i V_i)\beta_{ji}) + c_j(\partial_j^3\beta_{ij} - 2V_j\partial_j\beta_{ij} - (\partial_j V_j)\beta_{ij})\\
&\quad + \sum_{\substack{1\le p\le N,\\ p\neq i,j}} c_p(\beta_{ip}\partial_p^2\beta_{jp} + \beta_{jp}\partial_p^2\beta_{ip} - (\partial_p\beta_{ip})(\partial_p\beta_{jp}) - 2V_p\beta_{ip}\beta_{jp}) = 0,\\
&\partial_j V_i = 2\beta_{ij}\partial_i\beta_{ji} + \beta_{ji}\partial_i\beta_{ij}, \quad i\neq j.
\end{aligned}
\tag{27}
$$

For $N = 2$ they reduce to (8) under the identification $\beta_{12} = \beta$, $\beta_{21} = \gamma$. The kernel of D_3 is defined by the linear system

$$
\begin{aligned}
&\partial_i\Psi_j = \beta_{ij}\Psi_i,\\
&\partial_i^3\Psi_i = 2V_i\partial_i\Psi_i + (\partial_i V_i)\Psi_i - \sum_{\substack{1\le p\le N,\\ p\neq i}} \frac{c_p}{c_i}(\beta_{ip}\partial_p^2\Psi_p\\
&\quad -(\partial_p\beta_{ip})\partial_p\Psi_p + (\partial_p^2\beta_{ip} - 2V_p\beta_{ip})\Psi_p),
\end{aligned}
$$

which is compatible of rank $3N$ by virtue of the reduction equation (27). Each of the Ψ_i's can thus be regarded as a position vector of an N-dimensional submanifold M_i in P^{3N-1}, parametrized by conjugate coordinates. Moreover, the line (Ψ_i, Ψ_j) is tangential to both M_i and M_j. Introducing the quadratic form S by the equations

$$
\partial_i S = \Psi_i\Psi_i^*, \tag{28}
$$

one can readily show that this expression explicitly integrates to

$$
S = \sum_i c_i\left(\Psi_i\partial_i^2\Psi_i - \frac{1}{2}(\partial_i\Psi_i)^2 - V_i\Psi_i^2\right), \tag{29}
$$

thus defining the invariant symmetric scalar product on the kernel of D_3. One can show that all congruences (Ψ_i, Ψ_j) are isotropic with respect to S.

To the best of our knowledge, the geometric object consisting of N submanifolds in P^{3N-1}, $N \ge 3$, connected by isotropic congruences, has not been discussed before. In is not clear, for instance, whether such structure can be related to the Plücker image of an N-dimensional projective submanifold carrying a holonomic asymptotic net.

4. Reductions in the framework of multicomponent KP hierarchy

The main purposes of this section are:

- to demonstrate that the reduction (20) of multicomponent KP hierarchy leads to the existence of transformations (3), (14) used to characterize a class of reductions in geometric context;

- to explicitly construct the kernel of this transformations;
- to calculate generating potentials S for reductions of arbitrary order.

First we need to identify geometric objects (wave functions, potentials β_{ij}) in the framework of multicomponent KP hierarchy.

In our description of the infinite-dimensional Grassmannian we will follow Witten [36], considering the spaces W and W^* as dual spaces of boundary values of the operator $\bar\partial$, for both of which this operator has zero index. Slightly changing the standard setting for technical convenience, we will consider the problem of inversion of the $\bar\partial$-operator in the unit disc with center at zero (not at infinity), so that in the formula (17) and in the expression for $F(\lambda)$ one should change λ to λ^{-1}, $F_{ij}=\delta_{ij}F_i(\lambda)=\delta_{ij}c_i\lambda^{-n}$

$$g_i(\lambda)=g_i(\lambda,\mathbf{x})=\exp\left(\sum_{n=1}^{\infty}x_{i(n)}\lambda^{-n}\right). \tag{30}$$

The case when the operator $\bar\partial$ is invertible (has a kernel of zero dimension) for both spaces of boundary values, corresponds to the principal Grassmannian stratum. In this case both spaces W and W^* are transversal to the space of functions analytic in the unit disc. The evolution of a point belonging to the principal stratum is taking place in the principal stratum almost everywhere in $\mathbf{x}$, except a manifold of codimension one, where the objects we need for geometry (wave functions, potentials) have singularities. So we will consider the dynamics only on the principal stratum.

In the principal stratum, the inversion of the operator $(-2\pi\mathrm{i})^{-1}\bar\partial_\lambda$ with the space of boundary values $W(\lambda)$, and the inversion of the dual operator $2\pi\mathrm{i}\bar\partial_\mu$ with the space of boundary values $W^*(\mu)$, is defined by the same Green function (Cauchy kernel) $\chi(\lambda,\mu)$ [3], having very simple analytic properties: it is an $N\times N$ matrix function analytic in both variables in $\mathbf{D}$ outside $\lambda=\mu$, behaving as $(\lambda-\mu)^{-1}$ near $\lambda=\mu$. An arbitrary function with these properties defines a pair of dual points in the principal stratum W, W^*; the dynamics of the Cauchy kernel is characterized by Hirota's bilinear identity

$$\oint\chi(\nu,\mu;\mathbf{x})g(\nu,\mathbf{x})g^{-1}(\nu,\mathbf{x}')\chi(\lambda,\nu;\mathbf{x}')\,\mathrm{d}\nu=0, \tag{31}$$

implied by (15) and (16) (see [3]). Thus, the Cauchy kernel gives a compact representation of a dual pair of points on the Grassmannian, and defines all objects considered in the geometric setting.

Linear Eqs. (1), (2), (12) and (13), as well as linear transformations defining the reductions (3) and (14) are connected with expansions of some functions belonging to W and W^* with respect to a special basis. To construct this basis in the spaces W and W^*, we start with the functions $\chi_i(\lambda;\mathbf{x})=\chi_{i\_}(\lambda,0;\mathbf{x})\in W(\lambda,\mathbf{x})$ and $\chi_i^*(\lambda;\mathbf{x})=\chi_{\_i}(0,\mu;\mathbf{x})\in W^*(\lambda,\mathbf{x})$, where $\chi_{i\_}$ and $\chi_{\_i}$ denote ith row and ith column of the matrix χ. Using the operators $\mathcal{D}_i$, $\mathcal{D}_i^*$, $1\le i\le N$

$$\mathcal{D}_iW_j(\lambda,\mathbf{x})=(\partial_i+\delta_{ij}\lambda^{-1})W_j,\qquad \mathcal{D}_i^*W_j^*(\lambda,\mathbf{x})=(\partial_i-\delta_{ij}\lambda^{-1})W_j^*,$$

possessing the property

$$\mathcal{D}_iW(\lambda,\mathbf{x})\subset W(\lambda,\mathbf{x}),\qquad \mathcal{D}_i^*W^*(\lambda,\mathbf{x})\subset W^*(\lambda,\mathbf{x})$$

(this property is readily checked using (16) and (30); in the $\bar\partial$-dressing method the ring of operators possessing this property is known as the Zakharov–Manakov ring [32]), we obtain a basis in the form

$$\mathcal{D}_i^n \chi_i \in W, \qquad \mathcal{D}_i^{*n} \chi_i^* \in W^*, \quad 0 \le n < \infty, \ \ 1 \le i \le N. \tag{32}$$

Expressing the functions $\mathcal{D}_i \chi_j(\lambda, \mathbf{x}) \in W$ and $\mathcal{D}_i^* \chi_j(\lambda, \mathbf{x}) \in W^*$, $i \neq j$, through this basis, we get

$$\mathcal{D}_i \chi_j(\lambda, \mathbf{x}) = \beta_{ij}(\mathbf{x}) \chi_i(\lambda, \mathbf{x}), \tag{33}$$

$$\mathcal{D}_i^* \chi_j(\lambda, \mathbf{x}) = \beta_{ji}(\mathbf{x}) \chi_i^*(\lambda, \mathbf{x}), \tag{34}$$

where

$$\beta_{ij}(\mathbf{x}) = \chi_{ji}(0, 0; \mathbf{x}).$$

In terms of Baker–Akhieser functions

$$\psi_i(\lambda, \mathbf{x}) = \chi_i(\lambda; \mathbf{x}) g(\lambda, \mathbf{x}) \in W_0(\lambda), \qquad \psi_i^*(\lambda, \mathbf{x}) = g^{-1}(\lambda, \mathbf{x}) \chi_i^*(\lambda; \mathbf{x}) \in W_0^*(\lambda),$$

the operators $\mathcal{D}_i$ and $\mathcal{D}_i^*$ act as usual differentiations, and (33), (34) imply

$$\partial_i \psi_j(\lambda, \mathbf{x}) = \beta_{ij}(\mathbf{x}) \psi_i(\lambda, \mathbf{x}), \tag{35}$$

$$\partial_i \psi_j^*(\lambda, \mathbf{x}) = \beta_{ji}(\mathbf{x}) \psi_i^*(\lambda, \mathbf{x}). \tag{36}$$

Introducing scalar wave functions

$$\Psi_i(\mathbf{x}) = \oint \psi_i(\lambda, \mathbf{x}) \rho(\lambda)\, \mathrm{d}\lambda, \tag{37}$$

$$\Psi_i^*(\mathbf{x}) = \oint \rho^*(\lambda) \psi_i^*(\lambda, \mathbf{x})\, \mathrm{d}\lambda, \tag{38}$$

where $\rho(\lambda)$ (column) and $\rho^*(\lambda)$ (row) are arbitrary weight functions, we readily derive linear Eqs. (12) and (13) from (35) and (36).

Similarly, let us define a reduction by the formula (20) and consider the expansion of $\chi_i^{*\mathrm{t}}(-\lambda; \mathbf{x}) F(\lambda) \in W(\lambda, \mathbf{x})$ in the basis (32)

$$\chi_i^{*\mathrm{t}}(-\lambda; \mathbf{x}) F(\lambda) = c_i \mathcal{D}_i^n \chi_i(\lambda, \mathbf{x}) + \sum_{j=1}^{N} \sum_{p=0}^{n-1} U_{jn}(\mathbf{x}) \mathcal{D}_j^p \chi_j(\lambda, \mathbf{x}),$$

in terms of dual wave functions

$$\Psi_i^{*\,\mathrm{red}}(\mathbf{x}) = \oint \psi_i^{*\mathrm{t}}(-\lambda; \mathbf{x}) F(\lambda) \rho(\lambda)\, \mathrm{d}\lambda, \tag{39}$$

which can be represented in the form (38) with

$$\rho^*(\lambda) = \rho^{*\,\mathrm{red}}(\lambda) = -\rho^{\mathrm{t}}(-\lambda) F(-\lambda).$$

In this way one arrives at the linear transformations (3) and (14) which were used to define reductions in geometric context.

Let us consider the kernel of linear transformations (3), (14). This kernel is connected with the linear space of weight functions $\rho(\lambda)$, for which the dual wave functions $\Psi_i^{*\,\mathrm{red}}(\mathbf{x})$ are equal to zero

$$\Psi_i^{*\,\mathrm{red}}(\mathbf{x}) = \oint \psi_i^{*\mathrm{t}}(-\lambda;\mathbf{x}) F(\lambda)\rho(\lambda)\,\mathrm{d}\lambda = 0.$$

It is easy to conclude that

$$\rho(\lambda) \in F^{-1}(\lambda) W_0^{\mathrm{t}}(-\lambda).$$

Using the reduction condition (20) and some linear algebra, we represent the linear space $F^{-1}(\lambda)W_0^{\mathrm{t}}(-\lambda)$ in the form

$$F^{-1}(\lambda) W_0^{\mathrm{t}}(-\lambda) = W_0^*(\lambda) \oplus W_0^{*\mathrm{an}}(\lambda), \tag{40}$$

where $W_0^{*\mathrm{an}}(\lambda)$ is a finite-dimensional space of functions analytic in the unit disc. This space has the dimension $2n$ for the transformation (3), and Nn for the transformation (14) (n is the order of reduction, i.e., the power of λ^{-1} in $F(\lambda)$, and N is the number of components). The basis in this space is given by the functions

$$\rho_i^p(\lambda) = F^{-1}(\lambda)\partial_i^p \psi_i^{\mathrm{t}}(-\lambda,\mathbf{x})_{\mathbf{x}=\mathbf{0}} \in W_0^{*\mathrm{an}}(\lambda), \tag{41}$$

where $0 \le p \le n-1$. The representation (40) implies that wave functions belonging to the kernel of transformations (3) and (14)

$$\Psi_i(\mathbf{x}) = \oint \psi_i(\lambda;\mathbf{x})\rho(\lambda)\,\mathrm{d}\lambda, \quad \rho(\lambda) \in W_0^*(\lambda) \oplus W_0^{*\mathrm{an}}(\lambda),$$

correspond to the weight functions belonging to $W_0^{\mathrm{an}}(\lambda)$ (for $\rho(\lambda) \in W_0^*(\lambda)$ wave functions evidently equal zero), and the dimension of the kernel coincides with the dimension of the analytic space. An arbitrary wave function belonging to the kernel can be expressed as

$$\Psi_i(\mathbf{x}) = \oint \psi_i(\lambda,\mathbf{x}) \left(\sum_{k=1}^{N} \sum_{p=0}^{n-1} c_k^p \rho_k^p(\lambda) \right) \mathrm{d}\lambda, \tag{42}$$

where c_k^p are arbitrary constants, and the functions $\rho_i^p(\lambda)$ are given by the formula (41). Wave functions belonging to the kernel are specified by the values of the functions $\partial_i^p \Psi_i(\mathbf{x})$, $0 \le p \le n-1$, $1 \le i \le N$ at the initial point $\mathbf{x} = \mathbf{0}$ (i.e., these values define the constants in (42)). This is an immediate corollary of (3) and (14), or directly (42). One can demonstrate that the evolution of the vector of these values with respect to some time of the hierarchy is defined by a closed set of linear equations, and to explicitly express this vector through the initial vector for arbitrary values of times.

4.1. *Generating forms S for reductions of arbitrary order*

Now we will construct generating quadratic forms S for reductions of arbitrary order (see expressions (25) and (29) in second and third order cases). Generating forms explicitly define transformations (3) and (14) (in the odd order case these forms are defined by the formula (28), in the even order case this formula is replaced by (24)). An interesting feature is that for odd order reductions the forms are symmetric, while for even order reductions they are antisymmetric. This implies that reductions of even and odd orders are connected with geometric objects of different types, and an illustration of that is given by our detailed consideration of reductions of first, second and third orders.

In terms of the Cauchy kernel, the definition of reduction (20) leads to the equation

$$\oint \chi^{\mathrm{t}}(\lambda, -\nu; \mathbf{x}) F(\nu) \chi(\mu, \nu; \mathbf{x})\, \mathrm{d}\nu = 0, \quad \mu, \lambda \in \mathbf{D} \tag{43}$$

(equations of this type were used in [3] to define reductions in terms of the Cauchy kernel). The integration of this equation gives

$$\chi^{\mathrm{t}}(\lambda, -\mu; \mathbf{x}) F(\mu) - F(-\lambda) \chi(\mu, -\lambda; \mathbf{x}) = \mathrm{Res}_{\nu=0}(\chi^{\mathrm{t}}(\lambda, -\nu; \mathbf{x}) F(\nu) \chi(\mu, \nu; \mathbf{x})). \tag{44}$$

It is convenient to introduce another basis in the spaces W and W^* defined through the Cauchy kernel by the expansions

$$\chi_{i\_}(\lambda, \mu; \mathbf{x}) = \sum_{n=0}^{\infty} \chi_{i(n)}(\lambda, \mathbf{x}) \mu^n, \quad \chi_{i(n)}(\lambda, \mathbf{x}) \in W(\lambda, \mathbf{x}), \tag{45}$$

$$\chi_{\_i}(\lambda, \mu; \mathbf{x}) = \sum_{n=0}^{\infty} \chi^*_{i(n)}(\mu, \mathbf{x}) \lambda^n, \quad \chi^*_{i(n)}(\mu, \mathbf{x}) \in W^*(\mu, \mathbf{x}), \tag{46}$$

where $0 \le n < \infty$, $1 \le i \le N$ and $\chi_{i(0)}(\lambda, \mathbf{x}) = \chi_i(\lambda, \mathbf{x})$, $\chi^*_{i(0)}(\mu, \mathbf{x}) = \chi^*_i(\mu, \mathbf{x})$ in terms of notations introduced before. Then it is possible to transform the r.h.s. of Eq. (44) to

$$\mathrm{Res}_{\nu=0}(\chi^{\mathrm{t}}(\lambda, -\nu; \mathbf{x}) F(\nu) \chi(\mu, \nu; \mathbf{x})) = \sum_{i=1}^{N} c_i \sum_{p+q=n-1} \chi^{\mathrm{t}}_{i(p)}(\lambda, \mathbf{x}) (-1)^p \chi_{i(q)}(\mu, \mathbf{x}). \tag{47}$$

Performing a transition to scalar wave functions, we define the generating form S_n by the expression

$$S_n(\mathbf{x}) = \sum_{i=1}^{N} c_i S_n^i, \qquad S_n^i = \frac{1}{2} \sum_{p+q=n-1} \Psi'_{i(p)}(\mathbf{x}) (-1)^p \Psi_{i(q)}(\mathbf{x}), \tag{48}$$

where we have introduced higher wave functions

$$\Psi_{i(p)}(\mathbf{x}) = \oint \chi_{i(p)}(\lambda, 0; \mathbf{x}) g(\lambda, \mathbf{x}) \rho(\lambda)\, \mathrm{d}\lambda, \tag{49}$$

$\Psi_{i(0)}(\mathbf{x}) = \Psi_i(\mathbf{x})$; for $\Psi'_{i(n)}(\mathbf{x})$ the weight function is $\rho'(\lambda)$ (an arbitrary second weight function). Higher wave functions can be expressed through the wave functions Ψ_i, Ψ'_i and their derivatives by virtue of the formulae connecting the basises (32) and (45)

$$\Psi_{i(p+1)} = \partial_i \Psi_{i(p)} - \beta^p_{ii} \Psi_i, \tag{50}$$

$$\partial_j \Psi_{i(p)} = \beta_{ji}^p \Psi_j, \quad i \neq j, \; 0 \leq p < \infty. \tag{51}$$

Here $\beta_{ij}^0 = \beta_{ij}$, and higher potentials β_{ii}^p, β_{ji}^p are connected with β_{ij} by the relations implied by compatibility conditions for the relations (50) and (51)

$$\beta_{ji}^{p+1} = \partial_i \beta_{ji}^p - \beta_{ji}\beta_{ii}^p, \tag{52}$$

$$\partial_j \beta_{ii}^p = \beta_{ij}\beta_{ji}^p. \tag{53}$$

Using relations (50) recursively, it is possible to express S_n as a quadratic form in Ψ_i, Ψ_i' and their derivatives, with the coefficients connected with β_{ij} by the formulae (52) and (53) (and thus to get expressions of the type (25) and (29)). It is easy to see that the form (48) is symmetric for odd n and antisymmetric for even n, i.e., for odd n it is invariant under the permutation of Ψ_i and Ψ_i', and for even n the permutation changes the sign of the expression (the permutation of wave functions corresponds to the permutation of weight functions ρ, ρ').

To obtain the formula of the type (24) and (28), we will use the general equation

$$\partial_i \Phi(\mathbf{x}) = \Psi_i(\mathbf{x})\Psi_i^*(\mathbf{x}), \tag{54}$$

where $\Psi_i(\mathbf{x})$, $\Psi_i^*(\mathbf{x})$ are given by the expressions (37), (38), and

$$\Phi(\mathbf{x}) = \oint \rho^*(\mu) g^{-1}(\mu, \mathbf{x})\chi(\lambda, \mu; \mathbf{x}) g(\lambda, \mathbf{x})\rho(\lambda)\, \mathrm{d}\lambda\, \mathrm{d}\mu,$$

which follows directly from Hirota's bilinear identity (31) (see, e.g., [3]).

For odd n, from the formulae (44), (47) and (48) and the Eq. (54) we obtain

$$\partial_i S_n(\mathbf{x}) = \tfrac{1}{2}(\Psi_i \Psi_i^{*\,\mathrm{red}'} + \Psi_i' \Psi_i^{*\,\mathrm{red}}), \tag{55}$$

where $\Psi_i^{*\,\mathrm{red}}$ is defined by (39). Taking $\rho = \rho'$, we get exactly the formula (28). Thus, in the odd order case, the reduction is characterized by the Eq. (55), where the generating form is given by the expression (48). Using (52) and (53) one can verify that partial derivatives ∂_j of each of the terms S_n^i in the expression (48) can be represented as

$$\partial_i S_n^j = \tfrac{1}{2}(\Psi_i (D_n^j \Psi')_i + \Psi_i' (D_n^j \Psi)_i),$$

where D_n^j are linear differential operators. Then from the Eq. (55) we obtain the explicit formula for the transformations (3) and (14)

$$\Psi_i^{*\,\mathrm{red}} = \sum_{j=1}^{N} c_j (D_n^j \Psi)_i.$$

In the even order case instead of (55) we have

$$\partial_i S_n(\mathbf{x}) = \tfrac{1}{2}(\Psi_i \Psi_i^{*\,\mathrm{red}'} - \Psi_i' \Psi_i^{*\,\mathrm{red}}) = \Psi_i \wedge \Psi_i^{*\,\mathrm{red}}. \tag{56}$$

Partial derivatives ∂_j of the terms S_n^i in the expression (48) can be expressed as

$$\partial_i S_n^j = \Psi_i \wedge (D_n^j \Psi)_i.$$

The function $\Psi_i^{*\,\mathrm{red}}$ (respectively, transformations (3) and (14)), are defined by the Eq. (56) up to a term $U_i(\mathbf{x})\Psi_i$

$$\Psi_i^{*\,\mathrm{red}} = U_i(\mathbf{x})\Psi_i + \sum_{j=1}^{N} c_j (D_n^j \Psi)_i, \tag{57}$$

which can be fixed either by taking Baker–Akhiezer functions as wave functions and considering analytic properties in λ, or just by a direct substitution into the dual operator. Nevertheless, the *existence* of transformations (3) and (14) (given by the formula (57)), is implied by the Eq. (56), and can be used as a definition of the reduction.

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Appendix A. Surfaces in projective differential geometry

Projective differential geometry of surfaces M^2 in P^3 has been extensively developed in the first half of the 19th century in the works of Wilczynski, Fubini, Ĉech, Cartan, Tzitzeica, Demoulin, Rozet, Godeaux, Lane, Eisenhart, Finikov, Bol and many others. Based on [29] (see also [7–9,12]), let us briefly recall Wilczynski's approach to the construction of surfaces M^2 in projective space P^3 in terms of solutions of a linear system

$$\mathbf{r}_{xx} = \beta \mathbf{r}_y + \tfrac{1}{2}(V - \beta_y)\mathbf{r}, \qquad \mathbf{r}_{yy} = \gamma \mathbf{r}_x + \tfrac{1}{2}(W - \gamma_x)\mathbf{r}, \tag{A.1}$$

where β, γ, V, W are functions of x and y. If we cross-differentiate (A.1) and assume $\mathbf{r}, \mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_{xy}$ to be independent, we arrive at the compatibility conditions [21, p. 120]

$$\begin{gathered}\beta_{yyy} - 2\beta_y W - \beta W_y = \gamma_{xxx} - 2\gamma_x V - \gamma V_x, \qquad W_x = 2\gamma\beta_y + \beta\gamma_y, \\ V_y = 2\beta\gamma_x + \gamma\beta_x,\end{gathered} \tag{A.2}$$

which coincide with (9). For any fixed β, γ, V, W satisfying (A.2), the linear system (A.1) is compatible and possesses a solution $\mathbf{r} = (r^0, r^1, r^2, r^3)$ where $r^i(x, y)$ can be regarded as homogeneous coordinates of a surface in projective space P^3. One may think of M^2 as a surface in a three-dimensional space with position vector $\mathbf{R} = (r^1/r^0, r^2/r^0, r^3/r^0)$. If we choose any other solution $\tilde{\mathbf{r}} = (\tilde{r}^0, \tilde{r}^1, \tilde{r}^2, \tilde{r}^3)$ of the same system (A.1) then the corresponding surface $\tilde{M}^2$ with position vector $\tilde{\mathbf{R}} = (\tilde{r}^1/\tilde{r}^0, \tilde{r}^2/\tilde{r}^0, \tilde{r}^3/\tilde{r}^0)$ constitutes a projective transform of M^2, so that any fixed β, γ, V, W satisfying (A.2) define a surface M^2 uniquely up to projective equivalence. Moreover, a simple calculation yields

$$\mathbf{R}_{xx} = \beta \mathbf{R}_y + a\mathbf{R}_x, \qquad \mathbf{R}_{yy} = \gamma \mathbf{R}_x + b\mathbf{R}_y \tag{A.3}$$

($a = -2r_x^0/r^0, b = -2r_y^0/r^0$) which implies that x, y are asymptotic coordinates on M^2. System (A.3) can be viewed as an "affine gauge" of system (A.1). In what follows, we assume that our surfaces are hyperbolic and the corresponding asymptotic coordinates x, y are real.[1] Since Eq. (A.2) specify a surface uniquely up to projective equivalence, they can be viewed as the 'Gauss–Codazzi' equations in projective geometry.

Even though the coefficients β, γ, V, W define a surface M^2 uniquely up to projective equivalence via (A.1), it is not entirely correct to regard β, γ, V, W as projective invariants. Indeed, the asymptotic coordinates x, y are only defined up to an arbitrary reparametrization of the form

$$x^* = f(x), \qquad y^* = g(y), \tag{A.4}$$

which induces a scaling of the surface vector according to

$$\mathbf{r}^* = \sqrt{f'(x)g'(y)}\,\mathbf{r}. \tag{A.5}$$

Thus [4, p. 1], the form of Eq. (A.1) is preserved by the above transformation with the new coefficients $\beta^*, \gamma^*, V^*, W^*$ given by

$$\begin{aligned} &\beta^* = \frac{\beta g'}{(f')^2}, \qquad V^*(f')^2 = V + S(f), \qquad \gamma^* = \frac{\gamma f'}{(g')^2}, \\ &W^*(g')^2 = W + S(g), \end{aligned} \tag{A.6}$$

where $S(\cdot)$ is the Schwarzian derivative, that is

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The transformation formulae (A.6) imply that the symmetric two-form

$$2\beta\gamma\, \mathrm{d}x\, \mathrm{d}y,$$

and the conformal class of the cubic form

$$\beta\, \mathrm{d}x^3 + \gamma\, \mathrm{d}y^3$$

are absolute projective invariants. They are known as the projective metric and the Darboux cubic form, respectively, and play an important role in projective differential geometry. In particular, they define a 'generic' surface uniquely up to projective equivalence. The vanishing of the Darboux cubic form is characteristic for quadrics: indeed, in this case $\beta = \gamma = 0$ so that asymptotic curves of both families are straight lines. The vanishing of the projective metric (which is equivalent to either $\beta = 0$ or $\gamma = 0$) characterizes ruled surfaces. In what follows we exclude these two degenerate situations and require $\beta \neq 0$, $\gamma \neq 0$.

Using (A.4)–(A.6), one can verify that the four points

$$\begin{aligned} &\mathbf{r}, \qquad \mathbf{r}_1 = \mathbf{r}_x - \frac{1}{2}\frac{\gamma_x}{\gamma}\mathbf{r}, \qquad \mathbf{r}_2 = \mathbf{r}_y - \frac{1}{2}\frac{\beta_y}{\beta}\mathbf{r}, \\ &\boldsymbol{\eta} = \mathbf{r}_{xy} - \frac{1}{2}\frac{\gamma_x}{\gamma}\mathbf{r}_y - \frac{1}{2}\frac{\beta_y}{\beta}\mathbf{r}_x + \left(\frac{1}{4}\frac{\beta_y\gamma_x}{\beta\gamma} - \frac{1}{2}\beta\gamma\right)\mathbf{r} \end{aligned} \tag{A.7}$$

[1] The elliptic case is dealt with in an analogous manner by regarding x, y as complex conjugates.

are defined in an invariant way, that is, under the transformation formulae (A.4)–(A.6) they acquire a nonzero multiple which does not change them as points in projective space P^3. These points form the vertices of the so-called Wilczynski moving tetrahedron [4,12,29]. Since the lines passing through $\mathbf{r}, \mathbf{r}_1$ and $\mathbf{r}, \mathbf{r}_2$ are tangential to the x- and y-asymptotic curves, respectively, the three points $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2$ span the tangent plane of the surface M^2. The line through $\mathbf{r}_1, \mathbf{r}_2$ lying in the tangent plane is known as the directrix of Wilczynski of the second kind. The line through $\mathbf{r}, \boldsymbol{\eta}$ is transversal to M^2 and is known as the directrix of Wilczynski of the first kind. It plays the role of a projective 'normal'. Wilczynski's tetrahedron proves to be the most convenient tool in projective differential geometry.

Using (A.1) and (A.7), we easily derive for $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \boldsymbol{\eta}$ the linear equations [12, p. 42]

$$\begin{pmatrix} \mathbf{r} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \boldsymbol{\eta} \end{pmatrix}_x = \begin{pmatrix} \frac{1}{2}\frac{\gamma_x}{\gamma} & 1 & 0 & 0 \\ \frac{1}{2}b & -\frac{1}{2}\frac{\gamma_x}{\gamma} & \beta & 0 \\ \frac{1}{2}k & 0 & \frac{1}{2}\frac{\gamma_x}{\gamma} & 1 \\ \frac{1}{2}\beta a & \frac{1}{2}k & \frac{1}{2}b & -\frac{1}{2}\frac{\gamma_x}{\gamma} \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \boldsymbol{\eta} \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{r} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \boldsymbol{\eta} \end{pmatrix}_y = \begin{pmatrix} \frac{1}{2}\frac{\beta_y}{\beta} & 0 & 1 & 0 \\ \frac{1}{2}l & \frac{1}{2}\frac{\beta_y}{\beta} & 0 & 1 \\ \frac{1}{2}a & \gamma & -\frac{1}{2}\frac{\beta_y}{\beta} & 0 \\ \frac{1}{2}\gamma b & \frac{1}{2}a & \frac{1}{2}l & -\frac{1}{2}\frac{\beta_y}{\beta} \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ \mathbf{r}_1 \\ \mathbf{r}_2 \\ \boldsymbol{\eta} \end{pmatrix}, \tag{A.8}$$

where we introduced the notation

$$k = \beta\gamma - (\ln\beta)_{xy}, \qquad l = \beta\gamma - (\ln\gamma)_{xy}, \qquad a = W - (\ln\beta)_{yy} - \tfrac{1}{2}(\ln\beta)_y^2,$$
$$b = V - (\ln\gamma)_{xx} - \tfrac{1}{2}(\ln\gamma)_x^2. \tag{A.9}$$

The compatibility conditions of Eq. (A.8) imply

$$(\ln\beta)_{xy} = \beta\gamma - k, \qquad (\ln\gamma)_{xy} = \beta\gamma - l, \qquad a_x = k_y + \frac{\beta_y}{\beta}k, \qquad b_y = l_x + \frac{\gamma_x}{\gamma}l,$$
$$\beta a_y + 2a\beta_y = \gamma b_x + 2b\gamma_x, \tag{A.10}$$

which is just the equivalent form of the projective "Gauss–Codazzi" equation (A.2).

Eq. (A.8) can be rewritten in the Plücker coordinates. For a convenience of the reader we briefly recall this construction. Let us consider a line l in P^3 passing through the points $\mathbf{a}$ and $\mathbf{b}$ with the homogeneous coordinates $\mathbf{a} = (a^0 : a^1 : a^2 : a^3)$ and $\mathbf{b} = (b^0 : b^1 : b^2 : b^3)$.

With the line l we associate a point $\mathbf{a}\wedge\mathbf{b}$ in projective space P^5 with the homogeneous coordinates

$$\mathbf{a}\wedge\mathbf{b}=(p_{01}:p_{02}:p_{03}:p_{23}:p_{31}:p_{12}),$$

where

$$p_{ij}=\det\begin{pmatrix} a^i & a^j \\ b^i & b^j \end{pmatrix}.$$

The coordinates p_{ij} satisfy the well-known quadratic Plücker relation

$$p_{01}p_{23}+p_{02}p_{31}+p_{03}p_{12}=0. \tag{A.11}$$

Instead of $\mathbf{a}$ and $\mathbf{b}$ we may consider an arbitrary linear combinations thereof without changing $\mathbf{a}\wedge\mathbf{b}$ as a point in P^5. Hence, we arrive at the well-defined Plücker correspondence $l(\mathbf{a},\mathbf{b}) \to \mathbf{a}\wedge\mathbf{b}$ between lines in P^3 and points on the Plücker quadric in P^5. Plücker correspondence plays an important role in the projective differential geometry of surfaces and often sheds some new light on those properties of surfaces which are not 'visible' in P^3 but acquire a precise geometric meaning only in P^5. Thus, let us consider a surface $M^2 \in P^3$ with the Wilczynski tetrahedron $\mathbf{r},\mathbf{r}_1,\mathbf{r}_2,\boldsymbol{\eta}$ satisfying Eq. (A.8). Since the two pairs of points $\mathbf{r},\mathbf{r}_1$ and $\mathbf{r},\mathbf{r}_2$ generate two lines in P^3 which are tangential to the x- and y-asymptotic curves, respectively, the formulae

$$\mathcal{U}=\mathbf{r}\wedge\mathbf{r}_1, \qquad \mathcal{V}=\mathbf{r}\wedge\mathbf{r}_2,$$

define the images of these lines under the Plücker embedding. Hence, with any surface $M^2 \in P^3$ there are canonically associated two surfaces $\mathcal{U}(x,y)$ and $\mathcal{V}(x,y)$ in P^5 lying on the Plücker quadric (A.11). In view of the formulae

$$\mathcal{U}_x=\beta\mathcal{V}, \qquad \mathcal{V}_y=\gamma\mathcal{U},$$

we conclude that the line in P^5 passing through a pair of points $(\mathcal{U},\mathcal{V})$ can also be generated by the pair of points $(\mathcal{U},\mathcal{U}_x)$ (and hence is tangential to the x-coordinate line on the surface $\mathcal{U}$) or by a pair of points $(\mathcal{V},\mathcal{V}_y)$ (and hence is tangential to the y-coordinate line on the surface $\mathcal{V}$). Consequently, the surfaces $\mathcal{U}$ and $\mathcal{V}$ are two focal surfaces of the congruence of straight lines $(\mathcal{U},\mathcal{V})$ or, equivalently, $\mathcal{V}$ is the Laplace transform of $\mathcal{U}$ with respect to x and $\mathcal{U}$ is the Laplace transform of $\mathcal{V}$ with respect to y.

Introducing

$$\mathcal{A}=\mathbf{r}_2\wedge\mathbf{r}_1+\mathbf{r}\wedge\boldsymbol{\eta}, \quad \mathcal{B}=\mathbf{r}_1\wedge\mathbf{r}_2+\mathbf{r}\wedge\boldsymbol{\eta}, \qquad \mathcal{P}=2\mathbf{r}_2\wedge\boldsymbol{\eta}, \quad \mathcal{Q}=2\mathbf{r}_1\wedge\boldsymbol{\eta},$$

we arrive at the following equations for the Plücker coordinates:

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 & \beta & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -\beta a & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\gamma_x}{\gamma} & 1 & 0 \\ 0 & 0 & 0 & b & 0 & 1 \\ -\beta a & 0 & \beta & 0 & b & -\dfrac{\gamma_x}{\gamma} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_y = \begin{pmatrix} \dfrac{\beta_y}{\beta} & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 \\ 0 & a & -\dfrac{\beta_y}{\beta} & -\gamma b & 0 & \gamma \\ \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ -\gamma b & 0 & 0 & 0 & l & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}. \tag{A.12}$$

Eq. (A.12) are consistent with the following table of scalar products:

$$(\mathcal{U}, \mathcal{P}) = -1, \qquad (\mathcal{A}, \mathcal{A}) = 1, \qquad (\mathcal{V}, \mathcal{Q}) = 1, \qquad (\mathcal{B}, \mathcal{B}) = -1, \tag{A.13}$$

all other scalar products being equal to zero. This defines a scalar product of the signature $(3, 3)$ which is the same as that of the quadratic form (A.11). Equivalently, one can say that the quadratic form

$$S = \mathcal{Q}\mathcal{V} - \mathcal{P}\mathcal{U} + \tfrac{1}{2}(\mathcal{A}^2 - \mathcal{B}^2)$$

is an integral of (A.12). The explicit form of S in terms of $\mathcal{U}$ and $\mathcal{V}$ is

$$S = \mathcal{V}_{xx}\mathcal{V} - \tfrac{1}{2}\mathcal{V}_x^2 - V\mathcal{V}^2 - (\mathcal{U}_{yy}\mathcal{U} - \tfrac{1}{2}\mathcal{U}_y^2 - W\mathcal{U}^2).$$

Notice that Eq. (A.12) and the expression for S identically coincide with (10) and (11) if one sets $\mathcal{U} = \Psi_2$, $\mathcal{V} = \Psi_1$, $c_1 = -c_2 = 1$.

Appendix B. Congruences *W*

There exists an important class of transformations in projective differential geometry which leave the system (A.1) form-invariant. These transformations are generated by congruences W, and require a solution of certain Dirac equation on the surface M^2. Here we briefly recall this construction following [6,12,16].

Let M^2 be a surface with the position vector $\mathbf{r}$ satisfying (A.1). With any pair of functions $\mathcal{U}$ and $\mathcal{V}$ solving the Dirac equation (1)

$$\mathcal{U}_x = \beta\mathcal{V}, \qquad \mathcal{V}_y = \gamma\mathcal{U}, \tag{B.1}$$

we associate a surface $\tilde{M}^2$ with the position vector $\mathbf{r}'$ given by the formula

$$\mathbf{r}' = \mathcal{V}\mathbf{r}_1 - \mathcal{U}\mathbf{r}_2 + \frac{1}{2}\left(\mathcal{V}\frac{\gamma_x}{\gamma} - \mathcal{U}\frac{\beta_y}{\beta} - \mathcal{V}_x + \mathcal{U}_y\right)\mathbf{r}. \tag{B.2}$$

In order to write down the equations for $\mathbf{r}'$, it is convenient to introduce certain quantities which are combinations of $\mathcal{U}$, $\mathcal{V}$ and their derivatives. First of all, we define $\mathcal{A}$ and $\mathcal{B}$ by the formulae

$$\mathcal{U}_y = \frac{\beta_y}{\beta}\mathcal{U} + \mathcal{A}, \qquad \mathcal{V}_x = \frac{\gamma_x}{\gamma}\mathcal{V} + \mathcal{B}$$

(in fact, we are copying Eq. (A.12) for the Plücker coordinates). The compatibility conditions $\mathcal{U}_{xy} = \mathcal{U}_{yx}$ and $\mathcal{V}_{xy} = \mathcal{V}_{yx}$ imply

$$\mathcal{A}_x = k\mathcal{U}, \qquad \mathcal{B}_y = l\mathcal{V},$$

where l and k are the same as in (A.9). Let us introduce $\mathcal{P}$ and $\mathcal{Q}$ by the formulae

$$\mathcal{A}_y = a\mathcal{U} + \mathcal{P}, \qquad \mathcal{B}_x = b\mathcal{V} + \mathcal{Q}.$$

Then compatibility conditions imply

$$\mathcal{P}_x = -\beta a\mathcal{V} + k\mathcal{A}, \qquad \mathcal{Q}_y = -\gamma b\mathcal{U} + l\mathcal{B}.$$

Finally, we introduce the quantities H and K via

$$\mathcal{P}_y = a\mathcal{A} - \frac{\beta_y}{\beta}\mathcal{P} - \gamma b\mathcal{V} + \gamma\mathcal{Q} - K, \qquad \mathcal{Q}_x = b\mathcal{B} - \frac{\gamma_x}{\gamma}\mathcal{Q} - \beta a\mathcal{U} + \beta\mathcal{P} + H,$$

so that compatibility conditions imply that H and K satisfy the equation dual to (B.1)

$$H_y = \beta K, \qquad K_x = \gamma H. \tag{B.3}$$

Equations for $\mathcal{U}$, $\mathcal{A}$, $\mathcal{P}$, $\mathcal{V}$, $\mathcal{B}$, $\mathcal{Q}$, H, K can be rewritten in matrix form

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -\beta a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\gamma_x}{\gamma} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 1 & 0 & 0 \\ -\beta a & 0 & \beta & 0 & b & -\dfrac{\gamma_x}{\gamma} & 1 & 0 \\ * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix}_y = \begin{pmatrix} \frac{\beta_y}{\beta} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & -\frac{\beta_y}{\beta} & -\gamma b & 0 & \gamma & 0 & -1 \\ \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 & 0 & 0 \\ -\gamma b & 0 & 0 & 0 & l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta \\ * & * & * & * & * & * & * & * \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \\ H \\ K \end{pmatrix}, \tag{B.4}$$

where the elements $*$ are not yet specified. Eq. (B.4) reduce to (A.12) under the reduction $H = K = 0$. In what follows we will also need the quantity

$$S = \mathcal{Q}\mathcal{V} - \mathcal{P}\mathcal{U} + \tfrac{1}{2}(\mathcal{A}^2 - \mathcal{B}^2), \tag{B.5}$$

which, in view of (B.4), satisfies the equations

$$S_x = \mathcal{V}H, \qquad S_y = \mathcal{U}K. \tag{B.6}$$

Remark. The explicit form of S in terms of $\mathcal{U}$ and $\mathcal{V}$ is

$$S = \mathcal{V}_{xx}\mathcal{V} - \tfrac{1}{2}\mathcal{V}_x^2 - V\mathcal{V}^2 - (\mathcal{U}_{yy}\mathcal{U} - \tfrac{1}{2}\mathcal{U}_y^2 - W\mathcal{U}^2),$$

so that

$$H = \mathcal{V}_{xxx} - 2V\mathcal{V}_x - V_x\mathcal{V} - \beta\mathcal{U}_{yy} + \beta_y\mathcal{U}_y + (2\beta W - \beta_{yy})\mathcal{U},$$
$$-K = \mathcal{U}_{yyy} - 2W\mathcal{U}_y - W_y\mathcal{U} - \gamma\mathcal{V}_{xx} + \gamma_x\mathcal{V}_x + (2\gamma V - \gamma_{xx})\mathcal{V}.$$

Notice that equations $H = K = 0$ identically coincide with (8) under the obvious identifications. In particular, H and K solve the adjoint linear problem (B.3).

Now a direct calculation gives:

$$\mathbf{r} = -2\frac{\mathcal{V}}{S}\mathbf{r}'_x - 2\frac{\mathcal{U}}{S}\mathbf{r}'_y + \frac{1}{S}\left(\mathcal{A} + \mathcal{B} + \frac{\gamma_x}{\gamma}\mathcal{V} + \frac{\beta_y}{\beta}\mathcal{U}\right)\mathbf{r}'. \tag{B.7}$$

Eqs. (B.2) and (B.7) imply that the line $\mathbf{r} \wedge \mathbf{r}'$ joining the corresponding points $\mathbf{r}$ and $\mathbf{r}'$ is tangential to both surfaces M^2 amd $\tilde{M}^2$, which are thus focal surfaces of the line congruence $\mathbf{r} \wedge \mathbf{r}'$. Moreover, the formulae

$$\mathbf{r}'_{xx} = \frac{S_x}{S}\mathbf{r}'_x + \left(\frac{S_x}{S}\frac{\mathcal{U}}{\mathcal{V}} - \beta\right)\mathbf{r}'_y + \frac{1}{2}\left(V + \beta_y - \frac{S_x}{S\mathcal{V}}\left(\mathcal{A} + \mathcal{B} + \frac{\gamma_x}{\gamma}\mathcal{V} + \frac{\beta_y}{\beta}\mathcal{U}\right)\right)\mathbf{r}',$$
$$\mathbf{r}'_{yy} = \frac{S_y}{S}\mathbf{r}'_y + \left(\frac{S_y}{S}\frac{\mathcal{V}}{\mathcal{U}} - \gamma\right)\mathbf{r}'_x + \frac{1}{2}\left(W + \gamma_x - \frac{S_y}{S\mathcal{U}}\left(\mathcal{A} + \mathcal{B} + \frac{\gamma_x}{\gamma}\mathcal{V} + \frac{\beta_y}{\beta}\mathcal{U}\right)\right)\mathbf{r}' \tag{B.8}$$

(which are the result of quite a long calculation) demonstrate that x and y are asymptotic coordinates on the transformed surface $\tilde{M}^2$ as well, so that the congruence $\mathbf{r} \wedge \mathbf{r}'$ preserves the asymptotic parametrization of its focal surfaces. Such congruences play a central role in projective differential geometry and are known as congruences W. By a construction, a congruence W with one given focal surface M^2 is uniquely determined by a solution $\mathcal{U}$, $\mathcal{V}$ of the linear Dirac equation (B.1). Normalizing the vector $\mathbf{r}'$ as $\mathbf{r}' = \sqrt{S}\tilde{\mathbf{r}}$, we can rewrite Eq. (B.8) in the canonical form (A.1)

$$\tilde{\mathbf{r}}_{xx} = \tilde{\beta}\tilde{\mathbf{r}}_y + \tfrac{1}{2}(\tilde{V} - \tilde{\beta}_y)\tilde{\mathbf{r}}, \qquad \tilde{\mathbf{r}}_{yy} = \tilde{\gamma}\tilde{\mathbf{r}}_x + \tfrac{1}{2}(\tilde{W} - \tilde{\gamma}_x)\tilde{\mathbf{r}}, \tag{B.9}$$

where the transformed coefficients $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{V}$, $\tilde{W}$ are given by the formulae

$$\tilde{\beta} = \frac{S_x}{S}\frac{\mathcal{U}}{\mathcal{V}} - \beta = \frac{H\mathcal{U}}{S} - \beta, \qquad \tilde{\gamma} = \frac{S_y}{S}\frac{\mathcal{V}}{\mathcal{U}} - \gamma = \frac{K\mathcal{V}}{S} - \gamma,$$
$$\tilde{V} = V - \frac{S_x}{S}\frac{\mathcal{V}_x}{\mathcal{V}} + \frac{3}{2}\left(\frac{S_x}{S}\right)^2 - \frac{S_{xx}}{S}, \qquad \tilde{W} = W - \frac{S_y}{S}\frac{\mathcal{U}_y}{\mathcal{U}} + \frac{3}{2}\left(\frac{S_y}{S}\right)^2 - \frac{S_{yy}}{S}, \tag{B.10}$$

we point out the simple identity $\tilde{\beta}\tilde{\gamma} = \beta\gamma - (\ln S)_{xy}$. The surface $\tilde{M}^2$ is called a W-transform of the surface M^2. The construction of W-congruences on the transformed surface $\tilde{M}^2$ requires a solution of the transformed Dirac equation

$$\partial_x\tilde{\Psi}_2 = \tilde{\beta}\tilde{\Psi}_1, \qquad \partial_y\tilde{\Psi}_1 = \tilde{\gamma}\tilde{\Psi}_2, \tag{B.11}$$

where $\tilde{\beta}$ and $\tilde{\gamma}$ are given by (B.10). Let us introduce a potential M by the formulae

$$M_x = H\Psi_1, \qquad M_y = K\Psi_2,$$

the compatibility of which readily follows from (1) and (2). Then an arbitrary solution Ψ_1, Ψ_2 of (1) generates a solution $\tilde{\Psi}_1$, $\tilde{\Psi}_2$ of (B.11) by the formula

$$\tilde{\Psi}_1 = \Psi_1 - \frac{\mathcal{V}M}{S}, \qquad \tilde{\Psi}_2 = \Psi_2 - \frac{\mathcal{U}M}{S}, \tag{B.12}$$

which is a specialization of the Darboux–Levi transformation, see [25].

Congruences W provide a standard tool for constructing Bäcklund transformations. Suppose we are given a class of surfaces specified by certain extra constraints imposed on β, γ, V, W. Let us try to find a congruence W such that the second focal surface will also belong to the same class. This requirement imposes additional restrictions on $\mathcal{U}$ and $\mathcal{V}$, which usually turn to be linear and, moreover, contain an arbitrary constant parameter, so that Eq. (B.4) become a "Lax pair" for the class of surfaces under study. Since the Dirac equation (B.1) is a part of this Lax pair, it is not surprising that surfaces in projective differential geometry are closely related to the DS hierarchy. Particularly interesting classes of surfaces correspond to reductions which are quite familiar in the modern soliton theory. These are

isothermally-asymptotic surfaces ($\beta = \gamma$),
surfaces R_0 ($\beta = 1$ or $\gamma = 1$),
surfaces R ($\beta_y = \gamma_x$),
surfaces of Jonas ($\beta_x = \gamma_y$), etc.

The construction of the corresponding Bäcklund transformations was carried out primarily by Jonas [16], see also [7–9].

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